Circular Statistics: Analysis of Orientation and Time Data

Circular statistics are useful when trying to analyze data that are taken from a distribution that circles back on itself. For instance, consider the circular and linear distributions of 360 equidistant points. In the linear distribution 0 and 360 are at opposite ends of the distribution, whereas in the circular distribution the points 0 and 360 are one and the same.

To describe a circular distribution we use a ***mean vector*** designated with a magnitude or length, **r**, and a direction, Θ. To calculate the mean vector from a sample of data points (for instance, azimuths), first calculate the average x and y components of the mean vector as follows:

$$\bar{x} = \frac{1}{n}\sum_{i-1}^{n}\sin(\alpha_i) = \frac{1}{n}\left[\sin(\alpha_1) + \sin(\alpha_2) + \sin(\alpha_3) + \ldots \sin(\alpha_n)\right]$$

$$\bar{y} = \frac{1}{n}\sum_{i-1}^{n}\cos(\alpha_i) = \frac{1}{n}\left[\cos(\alpha_1) + \cos(\alpha_2) + \cos(\alpha_3) + \ldots \cos(\alpha_n)\right]$$

where n is the number of observations and α_i is the i^{th} azimuth or observation. The length or magnitude of the mean vector is :

$$r = \sqrt{\bar{x}^2 + \bar{y}^2}$$

The magnitude of the mean vector gives an indication of the relative dispersion or concentration of the observations (azimuths). The range of r is 0.0-1.0. If the data are clumped in one direction, then r will be near 1.0. If the data are randomly distributed, then r will be small (near 0.0).

To obtain the angle of the mean direction solve the following equations.

$$\sin(\Theta) = \frac{\bar{x}}{r} \Rightarrow \Theta = \sin^{-1}\left(\frac{\bar{x}}{r}\right) \qquad \text{AND} \qquad \cos(\Theta) = \frac{\bar{y}}{r} \Rightarrow \Theta = \cos^{-1}\left(\frac{\bar{y}}{r}\right)$$

There is only a single solution between 0 and 359 that satisfies both equations. The correct angle can be determined by looking at whether x is positive (+) or negative (-).

x	use this equation
+	$\Theta = \cos^{-1}\left(\frac{\bar{y}}{r}\right)$
-	$\Theta = 360 - \cos^{-1}\left(\frac{\bar{y}}{r}\right)$

RAYLEIGH TEST

The Rayleigh test is a statistical procedure for determining whether a circular distribution is random or non-random. That is, are the azimuths of a distribution clumped in a particular direction? Calculate a critical value (the test statistic), Z, for the Rayleigh test using the following formula:

$$Z = nr^2$$

where Z is the critical value, n is the number of observations or azimuths, and r is the magnitude of the mean vector (determined as above). The **null** (H_o) and **alternative** (H_a) hypotheses for the test are:

H_o = the bearings are randomly distributed.

H_a = the bearings are distributed nonrandomly.

Use Table B.34 (Zar 1999) to determine whether to **accept or reject** the null hypothesis. To use the Table find the number of azimuths in the column headed n. Read across the row until you find a Z value that is close to Z calculated for the data. If Z calculated > Z from Table B.34, then conclude that the probability (p) that the distribution is random is less than the value at the top of the column. For example, suppose we had 8 observations on homing pigeons and the calculated $Z = 2.95$. The calculated Z is between Z values 2.899 and 3.665 in the n=8 row of the table. We conclude that $0.05 > p > 0.02$. The probability that the pigeons were flying in random directions is between 5/100 and 2/100. Scientists generally reject the null hypothesis at $p<0.05$. In this example, we would reject the null hypothesis that the pigeons' bearings were randomly distributed. The bearings were significantly clumped.

V TEST

In some instances we would like to know if the sample is oriented in a particular (homeward) direction. The V test allows this comparison. The hypotheses are:

H_o = the bearings are randomly distributed with respect to the predicted direction.

H_a = the bearings are not randomly distributed with respect to the predicted direction.

First calculate $V' = n\bar{y}\cos(\Phi) + n\bar{x}\sin(\Phi)$, where Φ is the predicted homeward direction. Next calculate the critical value, $u = \frac{\sqrt{2}}{\sqrt{n}} V'$.

Now, use Table B.35 to find the probability that the direction is random with respect to the predicted direction. Suppose u=2.79 for 25 azimuths of migrating butterflies at a predicted direction of 135^o. From Table B.35 we find the calculated u between 2.772 and 3.038 for n=25 observations. We can reject the hypothesis because $0.00025>p>0.001$.

REFERENCES

Zar, J. H. 1999. *Biostatistical Analysis, 4th ed.* Prentice Hall, Upper Saddle River, NJ.

TABLE B.34 Critical Values of Rayleigh's z

n	α: 0.50	0.20	0.10	0.05	0.02	0.01	0.005	0.002	0.001
6	0.734	1.639	2.274	2.865	3.576	4.058	4.491	4.985	5.297
7	0.727	1.634	2.278	2.885	3.627	4.143	4.617	5.181	5.556
8	0.723	1.631	2.281	2.899	3.665	4.205	4.710	5.322	5.743
9	0.719	1.628	2.283	2.910	3.694	4.252	4.780	5.430	5.885
10	0.717	1.626	2.285	2.919	3.716	4.289	4.835	5.514	5.996
11	0.715	1.625	2.287	2.926	3.735	4.319	4.879	5.582	6.085
12	0.713	1.623	2.288	2.932	3.750	4.344	4.916	5.638	6.158
13	0.711	1.622	2.289	2.937	3.763	4.365	4.947	5.685	6.219
14	0.710	1.621	2.290	2.941	3.774	4.383	4.973	5.725	6.271
15	0.709	1.620	2.291	2.945	3.784	4.398	4.996	5.759	6.316
16	0.708	1.620	2.292	2.948	3.792	4.412	5.015	5.789	6.354
17	0.707	1.619	2.292	2.951	3.799	4.423	5.033	5.815	6.388
18	0.706	1.619	2.293	2.954	3.806	4.434	5.048	5.838	6.418
19	0.705	1.618	2.293	2.956	3.811	4.443	5.061	5.858	6.445
20	0.705	1.618	2.294	2.958	3.816	4.451	5.074	5.877	6.469
21	0.704	1.617	2.294	2.960	3.821	4.459	5.085	5.893	6.491
22	0.704	1.617	2.295	2.961	3.825	4.466	5.095	5.908	6.510
23	0.703	1.616	2.295	2.963	3.829	4.472	5.104	5.922	6.528
24	0.703	1.616	2.295	2.964	3.833	4.478	5.112	5.935	6.544
25	0.702	1.616	2.296	2.966	3.836	4.483	5.120	5.946	6.559
26	0.702	1.616	2.296	2.967	3.839	4.488	5.127	5.957	6.573
27	0.702	1.615	2.296	2.968	3.842	4.492	5.133	5.966	6.586
28	0.701	1.615	2.296	2.969	3.844	4.496	5.139	5.975	6.598
29	0.701	1.615	2.297	2.970	3.847	4.500	5.145	5.984	6.609
30	0.701	1.615	2.297	2.971	3.849	4.504	5.150	5.992	6.619
32	0.700	1.614	2.297	2.972	3.853	4.510	5.159	6.006	6.637
34	0.700	1.614	2.297	2.974	3.856	4.516	5.168	6.018	6.654
36	0.700	1.614	2.298	2.975	3.859	4.521	5.175	6.030	6.668
38	0.699	1.614	2.298	2.976	3.862	4.525	5.182	6.039	6.681
40	0.699	1.613	2.298	2.977	3.865	4.529	5.188	6.048	6.692
42	0.699	1.613	2.298	2.978	3.867	4.533	5.193	6.056	6.703
44	0.698	1.613	2.299	2.979	3.869	4.536	5.198	6.064	6.712
46	0.698	1.613	2.299	2.979	3.871	4.539	5.202	6.070	6.721
48	0.698	1.613	2.299	2.980	3.873	4.542	5.206	6.076	6.729
50	0.698	1.613	2.299	2.981	3.874	4.545	5.210	6.082	6.736
55	0.697	1.612	2.299	2.982	3.878	4.550	5.218	6.094	6.752
60	0.697	1.612	2.300	2.983	3.881	4.555	5.225	6.104	6.765
65	0.697	1.612	2.300	2.984	3.883	4.559	5.231	6.113	6.776
70	0.696	1.612	2.300	2.985	3.885	4.562	5.235	6.120	6.786
75	0.696	1.612	2.300	2.986	3.887	4.565	5.240	6.127	6.794
80	0.696	1.611	2.300	2.986	3.889	4.567	5.243	6.132	6.801
90	0.696	1.611	2.301	2.987	3.891	4.572	5.249	6.141	6.813
100	0.695	1.611	2.301	2.988	3.893	4.575	5.254	6.149	6.822
120	0.695	1.611	2.301	2.990	3.896	4.580	5.262	6.160	6.837
140	0.695	1.611	2.301	2.990	3.899	4.584	5.267	6.168	6.847
160	0.695	1.610	2.301	2.991	3.900	4.586	5.271	6.174	6.855
180	0.694	1.610	2.302	2.992	3.902	4.588	5.274	6.178	6.861
200	0.694	1.610	2.302	2.992	3.903	4.590	5.276	6.182	6.865
300	0.694	1.610	2.302	2.993	3.906	4.595	5.284	6.193	6.879
500	0.694	1.610	2.302	2.994	3.908	4.599	5.290	6.201	6.891
∞	0.6931	1.6094	2.3026	2.9957	3.9120	4.6052	5.2983	6.2146	6.9078

TABLE B.35 Critical Values of u for the V Test of Circular Uniformity

n	α: 0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
8	0.688	1.296	1.649	1.947	2.280	2.498	2.691	2.916	3.066
9	0.687	1.294	1.649	1.948	2.286	2.507	2.705	2.937	3.094
10	0.685	1.293	1.648	1.950	2.290	2.514	2.716	2.954	3.115
11	0.684	1.292	1.648	1.950	2.293	2.520	2.725	2.967	3.133
12	0.684	1.291	1.648	1.951	2.296	2.525	2.732	2.978	3.147
13	0.683	1.290	1.647	1.952	2.299	2.529	2.738	2.987	3.159
14	0.682	1.290	1.647	1.953	2.301	2.532	2.743	2.995	3.169
15	0.682	1.289	1.647	1.953	2.302	2.535	2.748	3.002	3.177
16	0.681	1.289	1.647	1.953	2.304	2.538	2.751	3.008	3.185
17	0.681	1.288	1.647	1.954	2.305	2.540	2.755	3.013	3.191
18	0.681	1.288	1.647	1.954	2.306	2.542	2.758	3.017	3.197
19	0.680	1.287	1.647	1.954	2.308	2.544	2.761	3.021	3.202
20	0.680	1.287	1.646	1.955	2.308	2.546	2.763	3.025	3.207
21	0.680	1.287	1.646	1.955	2.309	2.547	2.765	3.028	3.211
22	0.679	1.287	1.646	1.955	2.310	2.549	2.767	3.031	3.215
23	0.679	1.286	1.646	1.955	2.311	2.550	2.769	3.034	3.218
24	0.679	1.286	1.646	1.956	2.311	2.551	2.770	3.036	3.221
25	0.679	1.286	1.646	1.956	2.312	2.552	2.772	3.038	3.224
26	0.679	1.286	1.646	1.956	2.313	2.553	2.773	3.040	3.227
27	0.678	1.286	1.646	1.956	2.313	2.554	2.775	3.042	3.229
28	0.678	1.285	1.646	1.956	2.314	2.555	2.776	3.044	3.231
29	0.678	1.285	1.646	1.956	2.314	2.555	2.777	3.046	3.233
30	0.678	1.285	1.646	1.957	2.315	2.556	2.778	3.047	3.235
32	0.678	1.285	1.646	1.957	2.315	2.557	2.780	3.050	3.239
34	0.678	1.285	1.646	1.957	2.316	2.558	2.781	3.052	3.242
36	0.677	1.285	1.646	1.957	2.316	2.559	2.783	3.054	3.245
38	0.677	1.284	1.646	1.957	2.317	2.560	2.784	3.056	3.247
40	0.677	1.284	1.646	1.957	2.317	2.561	2.785	3.058	3.249
42	0.677	1.284	1.646	1.958	2.318	2.562	2.786	3.060	3.251
44	0.677	1.284	1.646	1.958	2.318	2.562	2.787	3.061	3.253
46	0.677	1.284	1.646	1.958	2.319	2.563	2.788	3.062	3.255
48	0.677	1.284	1.645	1.958	2.319	2.564	2.789	3.063	3.256
50	0.677	1.284	1.645	1.958	2.319	2.564	2.790	3.065	3.258
55	0.676	1.284	1.645	1.958	2.320	2.565	2.791	3.067	3.261
60	0.676	1.283	1.645	1.958	2.320	2.566	2.793	3.069	3.263
65	0.676	1.283	1.645	1.958	2.321	2.567	2.794	3.071	3.265
70	0.676	1.283	1.645	1.958	2.321	2.567	2.795	3.072	3.267
75	0.676	1.283	1.645	1.959	2.322	2.568	2.796	3.073	3.269
80	0.676	1.283	1.645	1.959	2.322	2.568	2.796	3.074	3.270
90	0.676	1.283	1.645	1.959	2.322	2.569	2.797	3.076	3.272
100	0.676	1.283	1.645	1.959	2.323	2.570	2.798	3.077	3.274
120	0.675	1.282	1.645	1.959	2.323	2.571	2.800	3.080	3.277
140	0.675	1.282	1.645	1.959	2.324	2.572	2.801	3.081	3.279
160	0.675	1.282	1.645	1.959	2.324	2.572	2.802	3.082	3.280
180	0.675	1.282	1.645	1.959	2.324	2.573	2.802	3.083	3.282
200	0.675	1.282	1.645	1.959	2.325	2.573	2.803	3.084	3.282
300	0.675	1.282	1.645	1.960	2.325	2.574	2.804	3.086	3.285
∞	0.6747	1.2818	1.6449	1.9598	2.3256	2.5747	2.8053	3.0877	3.2873